A

MAT 1320 A Fall 2009 November 11th, 8:30 Prof. Desjardins

TEST #2

Max = 20

Student Number: Solutions

- Time: 80 min.
- Only basic scientific calculators are permitted: non-programmable, non-graphing, no differentiation or integration capability. Notes or books are not permitted.
- Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.
- Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite. Graphs and sketches may be drawn in pencil.
- Problems require complete and clearly presented solutions and carry part marks if there is substantial correct work toward the solution.

A

1. [2 points] Find $\frac{dy}{dx}$ if $xy^2 + 3x^2y = xe^y$.

$$\frac{d}{dx}(xy^2) + \frac{d}{dx}(3x^2y) = \frac{d}{dx}(xe^y)$$

$$y^2 + 2xy\frac{dy}{dx} + 6xy + 3x^2\frac{dy}{dx} = e^y + xe^y\frac{dy}{dx}$$

$$(2xy + 3x^2 - xe^y)\frac{dy}{dx} = e^y - y^2 - 6xy$$

$$\boxed{\frac{dy}{dx} = \frac{e^y - y^2 - 6xy}{2xy + 3x^2 - xe^y}}$$

2. [4 points] Use logarithmic differentiation to find the derivative of $f(x) = \dfrac{x^2 \arcsin x}{(x^2+1)^3}$.

$$\ln(f(x)) = \ln\left(\frac{x^2 \arcsin x}{(x^2+1)^3}\right)$$

$$= \ln(x^2) + \ln(\arcsin x) - \ln((x^2+1)^3)$$

$$= 2\ln x + \ln(\arcsin x) - 3\ln(x^2+1)$$

then $\frac{d}{dx}(\ln(f(x))) = \frac{d}{dx}(2\ln x + \ln(\arcsin x) - 3\ln(x^2+1))$

$$\frac{1}{f(x)} f'(x) = \frac{2}{x} + \frac{1}{\arcsin x\sqrt{1-x^2}} - \frac{3(2x)}{x^2+1}$$

so $\boxed{f'(x) = \frac{x^2 \arcsin x}{(x^2+1)^3}\left[\frac{2}{x} + \frac{1}{\arcsin x\sqrt{1-x^2}} - \frac{6x}{x^2+1}\right]}$

A

3. [4 points] Find the following limits:

(a) $\lim_{x\to 0} \dfrac{\sin x - xe^x}{x^2}$ (b) $\lim_{x\to\infty} \left(1+\dfrac{3}{x}\right)^x$

a) $\lim_{x\to 0} \dfrac{\sin x - xe^x}{x^2} = \lim_{x\to 0} \dfrac{\cos x - e^x - xe^x}{2x}$

(0/0) still (0/0)

$= \lim_{x\to 0} \dfrac{-\sin x - e^x - e^x - xe^x}{2}$

$= \dfrac{-2}{2} = \boxed{-1}$

b, let $y = \left(1+\frac{3}{x}\right)^x$

then $\ln y = \ln\left(\left(1+\frac{3}{x}\right)^x\right) = x\ln\left(1+\frac{3}{x}\right)$

so $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x\ln\left(1+\frac{3}{x}\right)$ $(\infty\cdot 0)$

$= \lim_{x\to\infty} \dfrac{\ln(1+3/x)}{1/x}$ (0/0)

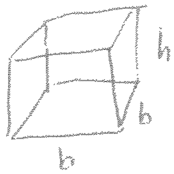
$= \lim_{x\to\infty} \dfrac{\frac{1}{1+\frac{3}{x}}\left(-3/x^2\right)}{\left(-1/x^2\right)}$

$= \lim_{x\to\infty} \dfrac{3}{1+3/x} = 3$

so $\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = \boxed{e^3}$

A

4. [4 points] If 2700 cm^2 of cardboard is available to make a box with an open top and a square base, find the largest possible volume of the box.



let b be the length of the side of the base and h the height

the surface area is $b^2 + 4bh = 2700$

the volume is $V = b^2 h$

but $h = \dfrac{2700 - b^2}{4b}$

so $V(b) = b^2\left(\dfrac{2700 - b^2}{4b}\right) = \dfrac{2700b - b^3}{4}$

thus $V'(b) = \dfrac{2700 - 3b^2}{4}$

$V'(b) = 0$ if $2700 - 3b^2 = 0$ or $b = 30$

if $b < 30$. $V'(b) > 0$
if $b > 30$ $V'(b) < 0$ so this is a max

(OR $V''(b) = -\frac{3}{2}b < 0 \Rightarrow$ max)

if $b = 30$ cm, $h = 15$ cm

and the max volume is $V = \boxed{13\,500 \text{ cm}^3}$

A

5. [6 points] Consider the function $y = f(x) = x + \dfrac{2}{x}$.

(i) Find any vertical or horizontal asymptotes.
(ii) Find the intervals of increase and decrease and any local extrema.
(iii) Find the intervals of concavity and any inflection points.
(iv) Use all of the information to sketch the graph.

i, $\lim_{x\to\infty}\left(x+\frac{2}{x}\right) = \infty$, $\lim_{x\to-\infty}\left(x+\frac{2}{x}\right) = -\infty$ $\boxed{\text{no horizontal asymptotes}}$

$f(x)$ undefined for $x=0$, $\lim_{x\to 0^+}\left(x+\frac{2}{x}\right)=\infty$, $\lim_{x\to 0^-}\left(x+\frac{2}{x}\right) = -\infty$

so $\boxed{x=0 \text{ is a vertical asymptote}}$

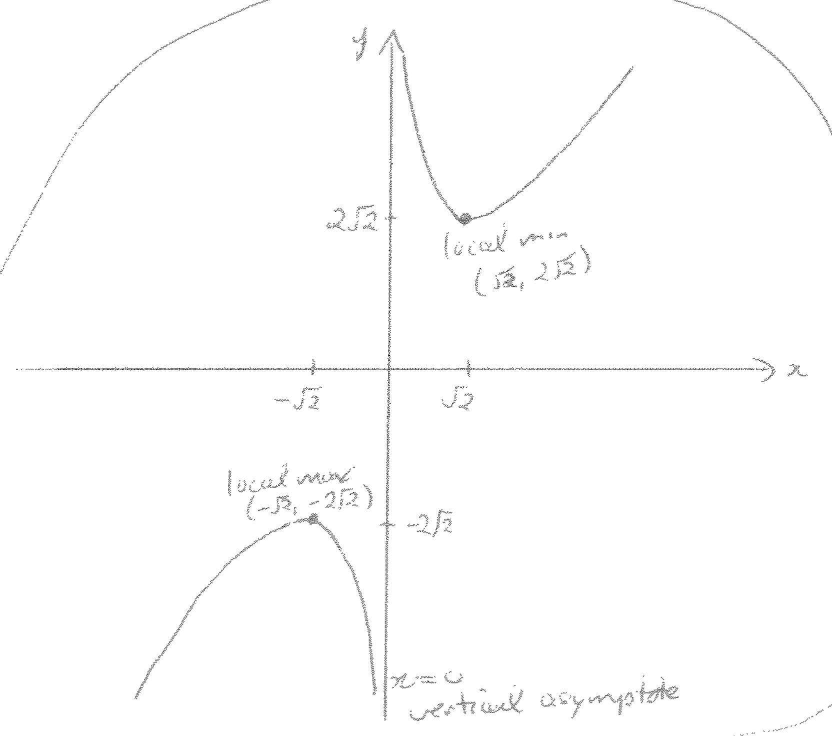
ii, $f'(x) = 1 - \dfrac{2}{x^2}$ $f'(x)=0$ if $x = \pm\sqrt{2}$

if $x < -\sqrt{2}$,	$f'(x) > 0$	$f(x)$ increasing	∴ local max at $(-\sqrt{2}, -2\sqrt{2})$
if $-\sqrt{2} < x < 0$,	$f'(x) < 0$	$f(x)$ decreasing	
if $0 < x < \sqrt{2}$,	$f'(x) < 0$	$f(x)$ decreasing	∴ local min at $(\sqrt{2}, 2\sqrt{2})$
if $x > \sqrt{2}$,	$f'(x) > 0$	$f(x)$ increasing	

iii, $f''(x) = \dfrac{4}{x^3}$ if $x<0$, $f''(x) < 0$; $f(x)$ concave down
if $x>0$; $f''(x) > 0$, $f(x)$ concave up

but no inflection points

iv,



B

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• Problems require complete and clearly presented solutions and carry part marks if there is substantial correct work toward the solution.

B

1. [2 points] Find $\frac{dy}{dx}$ if $2x^2y + xy^2 = xe^y$.

$$\frac{d}{dx}(2x^2y) + \frac{d}{dx}(xy^2) = \frac{d}{dx}(xe^y)$$

$$4xy + 2x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = e^y + xe^y\frac{dy}{dx}$$

$$(2x^2 + 2xy - xe^y)\frac{dy}{dx} = e^y - 4xy - y^2$$

$$\boxed{\frac{dy}{dx} = \frac{e^y - 4xy - y^2}{2x^2 + 2xy - xe^y}}$$

2. [4 points] Use logarithmic differentiation to find the derivative of $f(x) = \dfrac{x^3 \arctan x}{(x^2+2)^2}$.

$$\ln(f(x)) = \ln\left(\frac{x^3 \arctan x}{(x^2+2)^2}\right)$$

$$= \ln(x^3) + \ln(\arctan x) - \ln((x^2+2)^2)$$

$$= 3\ln x + \ln(\arctan x) - 2\ln(x^2+2)$$

so $$\frac{d}{dx}(\ln(f(x))) = \frac{d}{dx}(3\ln x + \ln(\arctan x) - 2\ln(x^2+2))$$

$$\frac{1}{f(x)} f'(x) = \frac{3}{x} + \frac{1}{(1+x^2)\arctan x} - \frac{2(2x)}{x^2+2}$$

so $$\boxed{f'(x) = \frac{x^3 \arctan x}{(x^2+2)^2}\left[\frac{3}{x} + \frac{1}{(1+x^2)\arctan x} - \frac{4x}{x^2+2}\right]}$$

B

3. [4 points] Find the following limits:

(a) $\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$ (b) $\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x$

a) $\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}$ still (0/0)

(0/0)

$= \lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x}$ still (0/0)

$= \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x}$

$= \frac{2}{1} = \boxed{2}$

b. let $y = \left(1 + \frac{2}{x}\right)^x$

then $\ln y = x \ln(1 + 2/x)$

so $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln(1+2/x)$ ($\infty \cdot 0$)

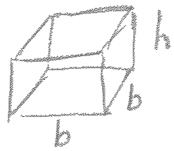
$= \lim_{x\to\infty} \frac{\ln(1+2/x)}{1/x}$ now (0/0)

$= \lim_{x\to\infty} \frac{\frac{1}{1+2/x}\left(-2/x^2\right)}{\left(-1/x^2\right)}$

$= \lim_{x\to\infty} \frac{2}{1+2/x} = 2$

So $\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^x = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = \boxed{e^2}$

4. [4 points] If 4800 cm^2 of cardboard is available to make a box with an open top and a square base, find the largest possible volume of the box.



let b be the length of the side of the base and h the height

surface area $\quad b^2 + 4hb = 4800 \Rightarrow h = \dfrac{4800 - b^2}{4b}$

the volume is $\quad V = b^2 h = b^2\left(\dfrac{4800 - b^2}{4b}\right) = \dfrac{4800b - b^3}{4}$

then $\quad V'(b) = \dfrac{4800 - 3b^2}{4}$

$V'(b) = 0$ if $b = 40$

if $b < 40$ $\quad V'(b) > 0$ $\quad$ so it's a max
if $b > 40$ $\quad V'(b) < 0$

(or $\quad V''(b) = -\frac{3}{2} b < 0 \Rightarrow$ max)

if $b = 40$ cm, $\quad h = 20$ cm

and the max volume is $\boxed{32\,000 \text{ cm}^3}$

B

5. [6 points] Consider the function $y = f(x) = x + \frac{3}{x}$.

(i) Find any vertical or horizontal asymptotes.
(ii) Find the intervals of increase and decrease and any local extrema.
(iii) Find the intervals of concavity and any inflection points.
(iv) Use all of the information to sketch the graph.

i, $\lim_{x\to\infty} \left(x + \frac{3}{x}\right) = \infty$, $\lim_{x\to-\infty} \left(x+\frac{3}{x}\right) = -\infty$ $\boxed{\text{so no horiz. asymp.}}$

$f(x)$ undefined at $x=0$, $\lim_{x\to 0^+} \left(x + \frac{3}{x}\right) = \infty$, $\lim_{x \to 0^-} \left(x+\frac{3}{x}\right) = -\infty$

so $\boxed{x=0 \text{ is vertical asymptote}}$

ii, $f'(x) = 1 - \frac{3}{x^2}$ $\quad f'(x) = 0$ if $x = \pm\sqrt{3}$

if $x < -\sqrt{3}$, $f'(x) > 0$, $f(x)$ increasing
if $-\sqrt{3} < x < 0$, $f'(x) < 0$, $f(x)$ decreasing
if $0 < x < \sqrt{3}$, $f'(x) < 0$, $f(x)$ decreasing
if $x > \sqrt{3}$, $f'(x) > 0$, $f(x)$ increasing

so local max at $(-\sqrt{3}, -2\sqrt{3})$

so local min at $(\sqrt{3}, 2\sqrt{3})$

iii, $f''(x) = \frac{6}{x^3}$, if $x<0$, $f''(x) < 0$, $f(x)$ concave down
if $x > 0$, $f''(x) > 0$, $f(x)$ concave up

but no inflection points

iv,